Bonaventura Cavalieri was an Italian mathematician who lived from 1598 to 1647. Cavalieri is well known for his work in geometry as well as optics and motion.

His first book dealt with the theory of mirrors shaped into parabolas, hyperbolas, and ellipses. What is most amazing about this work is that the technology to create the mirrors that he was writing about didn’t even exist yet!

Cavalieri is perhaps best known for his work with areas and volumes. He is so well known that he even has a principle named after him—Cavalieri’s principle.
PROBLEM 1  Approximating the Area of a Two-Dimensional Figure

One strategy for approximating the area of an irregularly shaped figure is to divide the figure into familiar shapes and determine the total area of all of the shapes. Consider the irregular shape shown. The distance across any part of the figure is the same.

1. You can approximate the area by dividing the irregular shape into congruent rectangles. To start, let's divide this shape into 10 congruent rectangles.

a. What is the length, the height, and the area of each congruent rectangle?

b. What is the approximate area of the irregularly shaped figure?

2. If this irregularly shaped figure were divided into 1000 congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?
3. If this irregularly shaped figure were divided into \( n \) congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?

4. If the irregularly shaped figure were divided into only one rectangle, what would be the approximate area of the figure?

5. Compare the area of the two figures shown. Each rectangle has a height of \( h \) and a base equal to length \( \ell \).

You have just explored **Cavalieri's principle** for two-dimensional figures, sometimes called the method of indivisibles. If the lengths of one-dimensional slices—just a line segment—of the two figures are the same, then the figures have the same area. This is best illustrated by making several slices to one figure and pushing them to the side to form a second figure.
Consider the right rectangular prism and the oblique rectangular prism shown.

1. What geometric figure represents a cross section of each that is perpendicular to the base?

2. What are the dimensions of one cross section?

3. What is the volume of the right rectangular prism?

4. Approximate the volume of the oblique rectangular prism by dividing the prism into ten congruent right prisms as shown.

Remember, a cross-section of a solid is the two-dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.
You have just explored Cavalieri’s principle for three-dimensional figures. Given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal. In other words, if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

For a second example of this principle, consider a right cylinder and an oblique cylinder having the same height and radii of equal length.

5. What geometric figure best represents the ten cross sections of the cylinders?

6. What are the dimensions of one cross section?

7. What is the volume of one cross section?

8. What is the volume of the oblique cylinder?

9. What is the volume of the right cylinder?

You have just shown the volume of a right cylinder and the volume of an oblique cylinder are equal, provided both cylinders have the same height and radii of equal length.
10. Using Cavalieri’s principle, what can you conclude about the volume of these two cones, assuming the heights are equal and the radii in each cone are congruent?

It is important to mention that the Cavalieri principles do not compute exact volumes or areas. These principles only show that volumes or areas are equal without computing the actual values. Cavalieri used this method to relate the area or volume of one unknown object to one or more objects for which the area or volume could be determined.

Talk the Talk

1. Consider the rectangle and the parallelogram shown to be of equal height with bases of the same length.

Knowing the area formula for a rectangle, how is Cavalieri’s principle used to determine the area formula for the parallelogram?

Be prepared to share your solutions and methods.